Lecture 3 (12/23/2021)

By induction, $\|D_v^{m+1} u\|_{L_2(Q_r)} \le N \|u\|_{L_2(Q_R)}$ $\quad \frac{1}{2} < r < R < 1$, $N(r, R)$

$m=0$ true for $1, 2, \cdots, m$, then for $m+1$

$\underline{u_t + v \cdot \nabla_x u - a_{ij} D_{v_i v_j} u = 0}$ $\qquad D_v u_t + v \cdot \nabla_x D_v u - a_{ij} D_{v_i v_j} D_v u = -D_x u$

$D_v^{m+1} u_t + v \nabla_x D_v^{m+1} u_t - a_{ij} D_{v_i v_j} D_v^{m+1} u = D_x D^m u$

$\|D_v^{m+2} u\|_{L_2(Q_r)} \le N\left(\|D_x D^m u\|_{L_2(Q_\tau)} + \|D^{m+1} u\|_{L_2(Q_\tau)}\right)$

$D_x u$ is also a solution. $\qquad \le N\|D_x u\|_{L_2(Q_{\tilde{\tau}})} + N\|u\|_{L_2(Q_R)} \le N\|u\|_{L_2(Q_R)}$

$D_x^l u$ is again a solution. $\quad \|D_v^m D_x^l u\|_{L_2} \le N \|D_x^l u\| \le N\|u\|_{L_2(Q_R)}$. $\quad$ Caccioppoli in x.

$\|\partial_t D_v^m D_x^l u\|_{L_2(Q_r)} \le N \|u\|_{L_2(Q_R)}$

Sobolev embedding, $\sup_t \|D_v^m D_x^l u\|_{L_2(B_{1/2} \times B_{1/2})} \le N \|u\|_{L_2(Q_R)}$

$$\Rightarrow \|D_v^m D_x^l u\|_{L_\infty} \le \text{RHS}. \qquad \|\partial_t D_v^m D_x^l u\|_{L_\infty} \le \text{RHS}. \quad \square$$

* **Cor**: $r > 0$, $\nu \ge 2$, $u \in S_{2,loc}$, $(P_0 + \lambda) u = 0$ in $(t_0 - (\nu r)^2, t_0) \times \mathbb{R}^d \times B_{\nu r}(x_0)$, $z_0 \in \mathbb{R}_T^{1+2d}$

Then $\left(|u - (u)_{Q_r(z_0)}|^2\right)^{1/2}_{Q_r(z_0)} \le N \nu^{-1} \left(|u|^2\right)^{1/2}_{Q_{\nu r}(z_0)}$

$\left(|(-\Delta_x)^{1/3} u - ((-\Delta_x)^{1/3} u)_{Q_r(z_0)}|^2\right)^{1/2}_{Q_r(z_0)} \le N \nu^{-1} \left(|(-\Delta_x)^{1/3} u|^2\right)^{1/2}_{Q_{2\nu r}(z_0)}$

$$\left(|D_v^2 u - (D_v^2 u)_{Q_r(z_0)}|^2\right)^{1/2}_{Q_r(z_0)} \le N \nu^{-1} \left(|D_v u|^2 + \lambda^2 u^2\right)^{1/2}_{Q_{\nu r}(z_0)} + N \nu^{-1} \sum_{k=0}^{\infty} 2^{-k} \left(|(-\Delta_x)^{1/3} u|^2\right)^{1/2}_{Q_{2\nu r, 2^k \nu r}}$$

Similar estimates for $D_v (-\Delta_x)^{1/6} u$.

Pf: $z_0 = 0$, $\nu r = 1$ $\qquad \nu^{-1} = r$

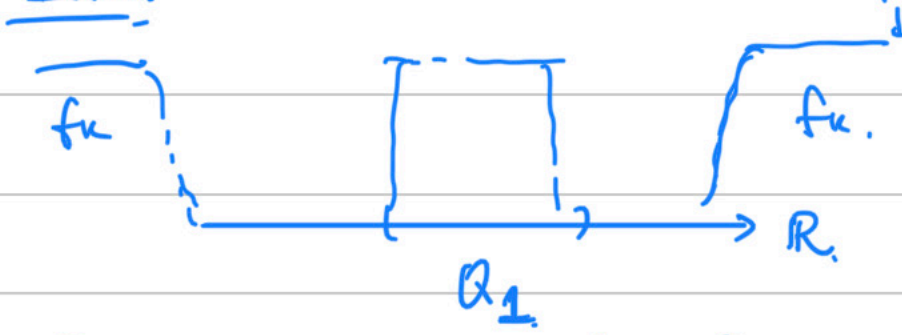
* **Lem 7**: $f \in L_2$ vanishes outside $(-1, 0) \times \mathbb{R}^d \times B_1$, $u \in S_2$ is a solution to the Cauchy problem
$$\begin{cases} P_0 u + \lambda u = f & \text{in} \\ u(-1, \cdot, \cdot) = 0. \end{cases} \qquad R \ge 1$$

$\left(t \in (-1, 0),\ v \in B_1,\ x \in B_{(2^k R)^3}\right)$

Then $\| |u| + |D_v u| + |D_v^2 u| \|_{L_2((-1,0) \times B_{R^3} \times B_R)} \le N \sum_{k=0}^{\infty} \underbrace{2^{-\frac{k(k-1)}{4}} R^{-k}}_{\text{fast decay}} \|f\|_{L_2(Q_{1, 2^k R})}$

$$\begin{cases} \left(|(-\Delta_x)^{1/3} u|^2\right)^{1/2}_{Q_{1,R}} \le N R^{-2} \sum_{k=0}^{\infty} 2^{-2k} \left(|f|^2\right)^{1/2}_{Q_{1, 2^k R}} \\ \left(|D_v (-\Delta_x)^{1/6} u|^2\right)^{1/2}_{Q_{1,R}} \le N R^{-1} \sum_{k=0}^{\infty} 2^{-k} \left(|f|^2\right)^{1/2}_{Q_{1, 2^k R}} \end{cases}$$

Idea:

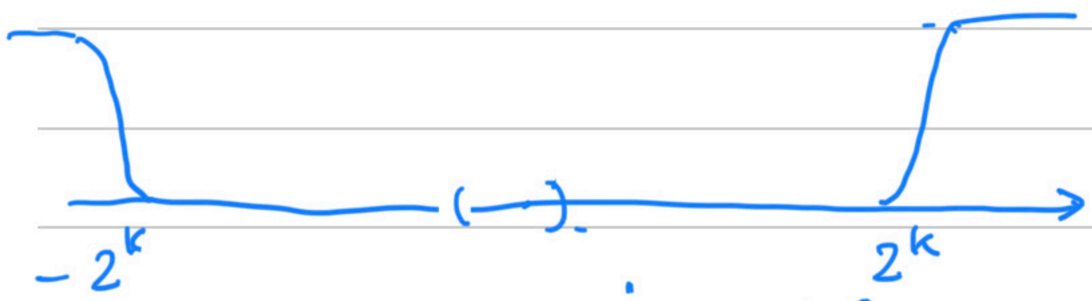


Decompose $f = \underbrace{f 1_{x \in B_{(2R)^3}}}_{= f_0} + \sum_{k=1}^{\infty} \underbrace{f 1_{B_{(2^{k+1} R)^3} \setminus B_{(2^k R)^3}}}_{= f_k}$.

For each k, solve the Cauchy problem $\to u_k$ $\qquad u = \sum_{k=0}^{\infty} u_k$

$\| |u_k| + |D_v u_k| + |D_v^2 u_k| \|_{L_2((-1,0) \times \mathbb{R}^d \times \mathbb{R}^d)} \le N \|f_k\|_{L_2(\cdots)}$ $\qquad$ (Ex)

Hint, change of variable (x, v)



…, change of variable

$\zeta_j^{(x,v)}$ for $j \le k-1$ $\;f_k \zeta_j \equiv 0$ in t, $\lambda \ge 1$.

Such that $\zeta_j \equiv 1$ in $B_{(2^{j+\frac12})^3} \times B_{2^{j+\frac12}}$

$\zeta_j \in C_0^\infty$ in $B_{(2^{j+1})^3} \times B_{2^j}$.

$|D_v \zeta_j| \le N 2^{-j}$, $|D_v^2 \zeta_j| \le N 2^{-2j}$, $|D_x \zeta_j| \le N 2^{-3j}$

Now. $P_0(u_k \zeta_j) + \lambda(u_k \zeta_j) = u_k P_0 \zeta_j - 2a D_v \zeta_j D_v u_k$

$$\Rightarrow \underbrace{\| |u_k| + |D_v u_k| + |D_v^2 u_k| \|_{L_2((-1,0)\times B_{2^{3j}} \times B_{2^j})}}_{A_j} \le N 2^{-j} \underbrace{\| |u_k| + |D_v u_k| \|_{L_2((-1,0)\times B_{2^{3(j+1)}} \times B_{2^{j+1}})}}_{A_{j+1}}$$

Iterate this inequality for $j = 0, 1, 2, \dots, k-1$.

$$\| \dots \|_{L_2((-1,0)\times B_1 \times B_1)} \le \underbrace{N^k 2^{-\frac{k(k-1)}{2}}}_{N 2^{-\frac{k(k-1)}{4}}} \underbrace{\| f_k \|_{L_2((-1,0)\times \mathbb{R}^{2d})}}_{\|f\|_{L_2(Q_{1,2^{k+1}})}}$$

$A_j \le N 2^{-j} A_{j+1}$

$A_0 \le N A_1 \le N \cdot 2^{-1} A_2 \le N 2^{-1-2} A_3 \le \dots$

$t \in (-1,0)$, $v \in B_1$, $x \in B_{(2^{k+1})^3}$

. $(-\Delta_x)^{1/3} u$, localize u. $\quad P_0(u\zeta_0) + \lambda u \zeta_0 = f\zeta_0 + u P_0 \zeta_0 - 2a D_v \zeta_0 D_v u$

L_2-estimate $\quad \|(-\Delta_x)^{1/3}(u\zeta_0)\|_{L_2} \le N \sum_{k=0}^\infty 2^{-\frac{k(k-1)}{4}} \|f\|_{L_2(Q_{1,2^{k+1}})}$

Need to estimate. $|(-\Delta_x)^{1/3}(u\zeta_0) - \zeta_0 (-\Delta_x)^{1/3} u| \le N \int_{|y|>CR^3} |u(t, x+y, v)| \, |y|^{-d-2/3} \, dy$

$$\le N \sum_{j=0}^\infty 2^{-2j - 3dj/2} \|u\|_{L_2(Q_{1,2^j})}$$

Reorder the summation. $\qquad$ use the 1st ineq.

Now we combine the corollary and Lem 7.

Prop. Let $r > 0$, $z_0 \in \mathbb{R}^{1+2d}_T$, $\nu \ge 2$. $u \in S_2(\mathbb{R}^{1+2d}_T)$. $P_0 u + \lambda u = f$ in $\mathbb{R}^{1+2d}_T$.

Then. $$\left(|(-\Delta_x)^{1/3} u - ((-\Delta_x)^{1/3} u)_{Q_r(z_0)}|^2\right)^{1/2}_{Q_r(z_0)}$$
$$\le N \nu^{-1} \left(|(-\Delta_x)^{1/3} u|^2\right)^{1/2}_{Q_{\nu r}(z_0)} + N \nu^{1+2d} \sum_{k=0}^\infty 2^{-2k} (f^2)_{Q_{2\nu r, 2^{k+1}\nu r}(z_0)}.$$

$$\left(|D_v^2 u - (D_v^2 u)_{Q_r(z_0)}|^2\right)^{1/2}_{Q_r(z_0)} \le N \nu^{-1} \left(|D_v^2 u|^2 + \lambda^2 u^2\right)^{1/2}_{Q_{\nu r}(z_0)} \qquad \text{(Ex).}$$
$$+ N \nu^{-1} \sum_{k=0}^\infty 2^{-k} \left(|(-\Delta_x)^{1/3} u|^2\right)^{1/2}_{Q_{\nu r, 2^k \nu r}(z_0)} + N \nu^{1+2d} \sum 2^{-k} (|f|^2)^{1/2}_{Q_{2\nu r, 2^k \nu r}(z_0)}$$

Similar estimate for λu and $D_v (-\Delta_x)^{1/6} u$.

Pf. $z_0 = 0$. $\nu r = 1$ $\quad \varphi(t, v) \in C_0^\infty((-4,4) \times B_2)$. $\varphi \equiv 1$ in $(-1,0) \times B_1$

$u = g + h$. $\quad \begin{cases} (P_0 + \lambda) g = f\varphi \\ g(-4, \cdot, \cdot) = 0 \end{cases}$, $\quad (P_0 + \lambda) h \equiv 0$ in $(-1,0) \times \mathbb{R}^d \times B_1$

then by Lem 7. $\quad (|(-\Delta_x)^{1/3} g|^2)^{1/2}_{Q_1} \le N \sum_{k=0}^\infty 2^{-2k} (|f|^2)^{1/2}_{Q_{2,2^{k+1}}}$ $\quad$ (V).

$(|(-\Delta_x)^{1/3} g|^2)^{1/2}_{Q_r} \le N \nu^{1+2d} \sum$

By Cor applied to h. $\left(|(-\Delta_x)^{1/3}h - ((-\Delta_x)^{1/3}h)_{Q_r}|^2\right)^{1/2}_{Q_r}$ $\quad 2 \to P$

$\le N \nu^{-1} \left(|(-\Delta_x)^{1/3} h|^2\right)^{1/2}_{Q_1} \le N\nu^{-1}\left(|(-\Delta_x)^{1/3}u|^2\right)^{1/2}_{Q_1}$

Now we bound $(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_r}$ $\quad + N\nu^{-1}\left(|(-\Delta_x)^{1/3}g|^2\right)^{1/2}_{Q_1}$

by using the Δ-ineq again.

Recall $(\mathbb{R}^{1+2d}_T, \hat{\rho}_c, dz)$ space of homogeneous type (quasi-metric with a doubling measure)

$\hat{\rho}_c(z, z_0) = \rho_c(z, z_0) + \rho_c(z_0, z)$. $\quad \rho_c(z, z_0) = \max\{|t-t_0|^{1/2}, |v-v_0|, c^{-1}|x - x_0 - v_0(t-t_0)|^{1/3}\}$

If $c \ge 1$, the constants in the quasi-Δ ineq and the doubling property are independent of c

Generalized maximal function. $M_c f(z_0) = \sup \fint_{Q_{r,cr}(z_1)} |f(z)|\,dz$. where $r > 0$, $z_0 \in Q_{r,cr}(z_1)$

$f^{\#}(z_0) = \sup \fint_{Q_r(z_1)} |f - (f)_{Q_r(z_1)}|\,dz$

Generalized Hardy-Littlewood thm: $\|M_c f\|_{L_p(\mathbb{R}^{1+2d}_T)} \le N\|f\|_{L_p(\mathbb{R}^{1+2d}_T)}$, N indep of c

Fefferman-Stein sharp function thm: $\|f\|_{L_p(\mathbb{R}^{1+2d}_T)} \le N \|f^{\#}\|_{L_p(\mathbb{R}^{1+2d}_T)}$

Need $f \in L_p$ (e.g. $f \equiv 1$).

By Prop, we have $((-\Delta_x)^{1/3}u)^{\#} \le N\nu^{-1} M^{1/2}|(-\Delta_x)^{1/3}u|^2$

$+ N\nu^{1+2d}\sum 2^{-2k} M^{1/2}_{2^k}(f^2)$. $\quad (*)$ at any point in $\mathbb{R}^{1+2d}_T$

$$\left[\ (D_v^2 u)^{\#} \le N\nu^{-1} M^{1/2}(|D_v^2 u|^2 + \lambda^2 u^2) + N\nu^{1+2d}\sum_{k=0}^{\infty} 2^{-k} M^{1/2}_{2^k} f^2 \quad + N\nu^{-1}\sum 2^{-k} M^{1/2}_{2^k}|(-\Delta_x)^{1/3}u|^2 \quad (**) \ \right] \quad 2 \to P.$$

Similar estimates for λu and $D_v(-\Delta_x)^{1/6}u$.

Thm: the result proved before is true if we replace 2 with $p \in (1, \infty)$.

Pf: $p > 2$. $\|(-\Delta_x)^{1/3}u\|_{L_p} \le \|((-\Delta_x)^{1/3}u)^{\#}\|_{L_p} \le N\nu^{-1}\|M^{1/2}|(-\Delta_x)^{1/3}u|^2\|_{L_p} + \dots$

$\|M(|(-\Delta_x)^{1/3}u|^2)\|^{1/2}_{L_{p/2}} \le \||(-\Delta_x)^{1/3}u|^2\|^{1/2}_{L_{p/2}} = \|(-\Delta_x)^{1/3}u\|_{L_p}$.

$\|D_v^2 u\|_{L_p} \le \|(D_v^2 u)^{\#}\|_{L_p} \le N\nu^{-1}(\|D_v^2 u\|_{L_p} + \lambda\|u\|_{L_p})$

$+ N\nu^{-1}\sum 2^{-k}\|(-\Delta_x)^{1/3}u\|_{L_p}$

$\|\lambda u\|_{L_p} \le$ similar term.

Solvability: localize the L_p estimate, $\Rightarrow (P_0 + \lambda)C_0^{\infty}$ is dense in $L_p \Rightarrow$ solvability

when $p \in (1, 2)$, we duality argument. (Need to be careful. $\partial_t u$, $\partial_x u$ are distributions)

$\partial_t u + v\cdot\nabla_x u \in L_p$

$a_{ij}(t)$.

Next, we deal with more general coefficients.

Denote. $\operatorname{osc}_{x,v}(a, Q_r(z_0)) = \fint_{t_0 - r^2}^{t_0} \fint_{D_r(z_0,t)\times D_r(z_0,t)} |a(t, x_1, v_1) - a(t, x_2, v_2)|\,dx_1\,dv_1\,dx_2\,dv_2$

$Q_r(z_0)$: Kinetic cylinder; the inner integral: γ_0

$D_r(z_0, t) = \{(x, v) : |v - v_0| < r,\ |x - x_0 - (t-t_0)v_0| < r^3\}$

Assumption: $\exists R_0>0$ s.t. $\forall r\in(0,R_0]$, $\forall z_0\in\mathbb{R}^{1+2d}$, $\mathrm{osc}_{x,v}(a, Q_r(z_0)) \le \gamma_0$ (small constant).

Stronger conditions: ① if a_{ij}'s are unif continuous in (x,v)

② If a_{ij} satisfies $\forall t_0$, $r\le R_0$, any v_0, x_0: $\fint_{B_{r^3}(x_0)\times B_r(v_0)} \fint_{B_{r^3}(x_0)\times B_r(v_0)} |a(t_0,x_1,v_1)-a(t_0,x_2,v_2)|\,dx_1dv_1dx_2dv_2 \le \gamma_0$

For the application of the boundary est / nonlinear problem, a_{ij}'s are continuous in t, but not unif continuous.

Thm: Suppose the assumption is satisfied. $p\in(1,\infty)$. Then $\exists \lambda_0\ge 1$ suff large s.t. the following is true for any $\lambda\ge\lambda_0$.

① If $u\in S_p(\mathbb{R}^{1+2d}_T)$ solves $(P+\lambda)u=f$ in $\mathbb{R}^{1+2d}_T$, then

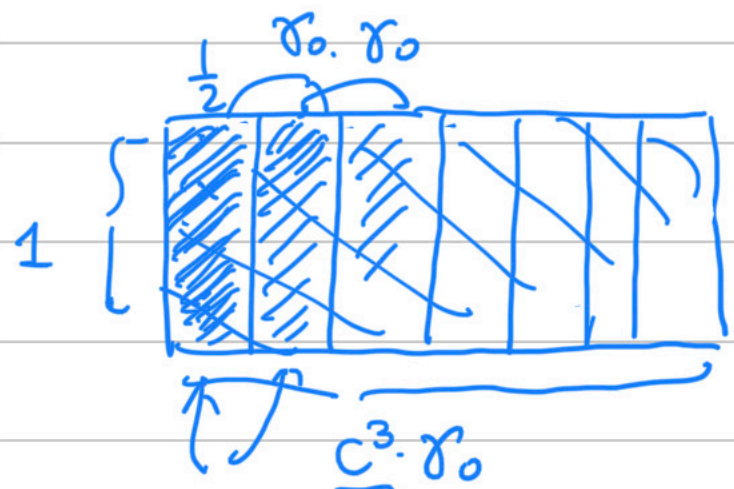
$$\left.\begin{aligned}&\lambda\|u\|_{L_p(\mathbb{R}^{1+2d}_T)} + \sqrt{\lambda}\|D_v u\|_{L_p} + \|D_v^2 u\|_{L_p} + \|(-\Delta_x)^{1/3}u\|_{L_p} + \|D_v(-\Delta_x)^{1/6}u\|_{L_p}\\ &\le N\|f\|_{L_p(\mathbb{R}^{1+2d}_T)}.\end{aligned}\right\}\ \text{A priori estimate}$$

② $\forall \lambda>\lambda_0$, $\exists!\ u\in S_p$ to this equation. $\Leftarrow$ Method of continuity

③ The Cauchy problem on $(0,T)$ with the zero initial condition has a unique sol. $(\lambda\ge 0)$. $\tilde u = u e^{\lambda_0 t}$

observation. $\forall c\ge 1$ $\fint_{Q_{r,cr}} |a - (a(t,\cdot,\cdot))_{B_{(cr)^3}\times B_r}|\,dz \le N c^3\gamma_0$

$r=1$



Mean oscillation estimates for equations with $VMO_{x,v}$ coefficients

$\forall r< R_0/4\nu$, $p_0>1$, $\alpha>1$ close to 1. (p is given).

Method of frozen coefficients. $u_t + v\cdot\nabla_x u - a_{ij}(t,x,v)D_{v_iv_j}u = f$

$\overline{a_{ij}}(t)$ is the average of a with respect to x,v in $Q_{2\nu r}$

Rewrite the equation into $u_t + v\cdot\nabla_x u - \overline{a_{ij}}(t)D_{v_iv_j}u = \underbrace{f + (a_{ij}-\overline{a_{ij}}(t))D_{v_iv_j}u}_{\tilde f}$

$$(|\tilde f|^{p_0})^{1/p_0}_{Q_{2\nu r,2^{k+1}\nu r}} \le (|f|^{p_0})^{1/p_0}_{Q_{2\nu r,2^{k+1}\nu r}}$$

$$+ (|a_{ij}-\overline{a_{ij}}(t)|\,|D_{v_iv_j}u|^{p_0})^{1/p_0}_{Q_{2\nu r,2^{k+1}\nu r}}$$

$$\le C(|D_v^2u|^{p_0\alpha})^{\frac{1}{p_0\alpha}}_{Q_{2\nu r,2^{k+1}\nu r}}\cdot (|a-\overline{a}(t)|^{\frac{p_0\alpha}{\alpha-1}})^{\frac{\alpha-1}{p_0\alpha}}$$

$$\le \underbrace{\left(|D_v^2 u|^{p_0\alpha}\right)^{\frac{1}{p_0\alpha}}_{Q_{22r,\,2^{k+1}2r}}}_{\text{Bound by the generalized Max function}} \cdot \underbrace{\left(|a-\bar{a}(t)|^{p_0\frac{\alpha}{\alpha-1}}\right)^{\frac{\quad}{p_0\alpha}}}$$

$$\le \left(|a-\bar{a}(t)|\right)^{\frac{\alpha-1}{p_0\alpha}}$$

$$\le N\left(2^{3K}\right)^{\boxed{\frac{\alpha-1}{p_0\alpha}}} \leftarrow \text{small}$$

Step 1: prove the estimate when u has a small support. $\underbrace{3\frac{\alpha-1}{p_0\alpha}}<1$. $p>p_0\alpha$.

Step 2: Use the partion of unity to extend this to the general case.

Then we

Next time: Boundary estimate (specular reflection boundary condition. $x\in\Omega$, on $\partial\Omega$).

Nonlinear equations.